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Th. If $o(G) = p^2$ where $p$ is a prime number, then $G$ is abelian.

Proof We shall show that the centre $Z$ of $G$ is equal to $G$ itself. Then obviously $G$ will be an abelian Group.

Since $p$ is a prime number, therefore by the ~~previous~~ Theorem if $o(G) = p^n$ where $p$ is a prime no. then the centre $Z \neq \{e\}$

i.e $Z \neq \{e\}$. Therefore $o(Z) > 1$.
But $Z$ is a Subgroup of $G$, therefore $o(Z)$ must be a divisor of $o(G)$
i.e. $o(Z)$ must be a divisor of $p^2$
Since $p$ is prime, therefore ~~there~~ either $o(Z) = p$ or $p^2$
If $o(Z) = p^2$ then $Z = G$ and
Proved

Now suppose that $o(Z) = p$. Then ~~$o(Z)$ must be a divisor of $p^2$~~
$o(Z) < o(G)$ because $p < p^2$
So there must be an elt which is in $G$ but which is not in $Z$

Let $a \in G$ and $a \notin Z$

Now $N(a)$ is a Subgroup of $G$ and $a \in N$

Also $x \in Z \Rightarrow xa = ax$ and this implies $x \in N(a)$. Thus $Z \subset N(a)$. Since $a \in Z$ therefore the number of elements in $N(a)$ is $> p$ i.e.

$o[N(a)] > p$. But order of $N(a)$ must be a divisor of $p^2$. Therefore $o[N(a)]$ must be equal to $p^2$. Then $N(a) = G$. Therefore $a \in Z$ and thus we get a contradiction

Therefore it is not possible that $o(Z) = p$

Hence the only posibility is that

$$o(Z) = p^2 \Rightarrow Z = G \Rightarrow G \text{ is abelian}$$